Math 10B with Professor Stankova
Worksheet, Discussion #19; Tuesday, 4/3/2018
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1 Partial Fractions

1.1 Concepts

1. Partial fractions allow us to compute an antiderivative of an expression of the form $P(x)/Q(x)$, where P, Q are polynomials, more easily (these are just fractions where the numerator and denominator are both fractions). First long divide so that the degree or highest term of the polynomial P is less than Q. Then factor $Q(x)$ into linear factors if you can, or else quadratic factors. Then for each factor, write the simplification of the form:

Factor	$ax+b$	$(ax+b)^n$	ax^2+bx+c	$(ax^2+bx+c)^n$
Expression	$\frac{A}{ax+b}$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots$	$\frac{Ax+B}{ax^2+bx+c}$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots$

Afterwards, find what these constants are. One good way to do this is to multiply everything by $Q(x)$ to clear denominators and then plug in different values of x.

1.2 Examples

2. Find $\int \frac{x^2}{x^2+3x-18} dx$.

Solution: First we have to long divide because the degrees are the same. We do this by writing $x^2 = (x^2+3x-18) + (x^2 - (x^2+3x-18)) = (x^2+3x-18) + (-3x+18)$ and so

$$\int \frac{x^2}{x^2+3x-18} dx = \int \frac{x^2+3x-18}{x^2+3x-18} + \frac{-3x+18}{x^2+3x-18} dx = \int 1 + \frac{-3x+18}{(x+6)(x-3)} dx.$$

Now we write $\frac{-3x+18}{(x+6)(x-3)} = \frac{A}{x+6} + \frac{B}{x-3}$. We solve for the constants by multiplying through by $(x+6)(x-3)$ to get $-3x+18 = A(x-3) + B(x+6)$. Finally, we solve for the constants by plugging in values for x. We can let $x = 3$ to get $9B = 9$ and $x = -6$ to get $-9A = 36$ so $B = 1$ and $A = -4$. Thus, we have

$$= \int 1 + \frac{1}{x-3} - \frac{4}{x+6} dx = x + \ln|x-3| - 4\ln|x+6| + C.$$

3. Find $\int \frac{x^3+3x^2+3x+3}{(x+1)^2(x^2+1)} dx$.

Solution: We split it as $\frac{x^3+3x^2+3x+3}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$. Multiplying through by $(x+1)^2(x^2+1)$ gives us

$$x^3 + 3x^2 + 3x + 3 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2.$$

Letting $x = -1, 0, 1, 2$ give us

$$2 = 2B, 3 = A + B + D, 10 = 4A + 2B + 4C + 4D, 29 = 15A + 5B + 18C + 9D.$$

Thus $B = 1$ and $A + D = 2$ so $4A + 2B + 4C + 4D = 10 + 4C = 10$ so $C = 0$. Finally, we have that $15A + 5 + 9D = 29$ so $A = B = D = 1$ and hence

$$\int \frac{x^3 + 3x^2 + 3x + 3}{(x+1)^2(x^2+1)} dx$$

$$= \int \frac{1}{x+1} + \frac{1}{(x+1)^2} + \frac{1}{x^2+1} dx$$

$$= \ln|x+1| - \frac{1}{x+1} + \arctan(x) + C.$$

1.3 Problems

4. True **FALSE** To find the partial fraction decomposition of $\frac{4x^3}{(x-1)(x+2)^2}$, we set it equal to $\frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ and solve for A, B, C.

Solution: The degrees are the same and so we first need to long divide before doing this.

5. Integrate $\int \frac{5x}{x^2-9x-36} dx$.

Solution: We have that $\frac{5x}{x^2-9x-36} = \frac{5x}{(x-12)(x+3)} = \frac{A}{x-12} + \frac{B}{x+3}$. Multiplying gives $5x = A(x+3) + B(x-12)$ and plugging in $x = -3$ and $x = 12$ gives $-15 = -15B$ and $60 = 15A$ respectively or $A = 4, B = 1$ and hence

$$\int \frac{5x}{x^2 - 9x - 36} dx = \int \frac{4}{x-12} + \frac{1}{x+3} dx = 4\ln|x-12| + \ln|x+3| + C.$$

6. Integrate $\int \frac{4x^2}{(x-1)(x-2)^2} dx$.

Solution: We set $\frac{4x^2}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$. Multiplying through gives us $4x^2 = A(x-2)^2 + B(x-1)(x-2) + C(x-1)$. Now set $x = 1$ to get $4 = A$ and $x = 2$ to get $C = 16$. Now plugging in 0 gives us $0 = 4A + 2B - C = 16 + 2B - 16 = 2B$ and so $B = 0$. Thus, we have

$$\int \frac{4x^2}{(x-1)(x-2)^2}dx = \int \frac{4}{x-1} + \frac{16}{(x-2)^2}dx = 4\ln|x-1| - \frac{16}{x-2} + C.$$

7. Set up the partial fraction decomposition of $\frac{8x^3+3x^2+1}{(x-1)^2(x^2+4)^2}$ (you don't have to solve for the coefficients).

Solution:

$$\frac{8x^3+3x^2+1}{(x-1)^2(x^2+4)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2}.$$

8. Integrate $\int \frac{\sec^2(x)}{\tan(x)^2 - \tan(x)}dx$.

Solution: First we u sub by letting $u = \tan(x)$ and then $du = \sec^2(x)dx$ so we have that this integral is $\int \frac{du}{u^2-u}$. Then we write $\frac{1}{u^2-u} = \frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$. Multiplying gives $A(u-1) + Bu = 1$ and letting $u = 1$ gives $B = 1$ and let $u = 0$ to get $-A = 1$ so $A = -1$. Thus

$$\int \frac{du}{u^2-u} = \int \frac{1}{u-1} - \frac{1}{u}du = \ln|u-1| - \ln|u| + C = \ln|\tan(x)-1| - \ln|\tan(x)| + C.$$

1.4 Extra Problems

9. Integrate $\int \frac{5x+17}{x^2+2x-15}dx$.

Solution: We have that $\frac{5x+17}{x^2+2x-15} = \frac{5x+17}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$. Multiplying gives $5x + 17 = A(x+5) + B(x-3)$ and plugging in $x = 3$ and $x = -5$ gives $32 = 8A$ and $-8 = -8B$ respectively or $A = 4, B = 1$ and hence

$$\int \frac{5x+17}{x^2+2x-15}dx = \int \frac{4}{x-3} + \frac{1}{x+5}dx = 4\ln|x-3| + \ln|x+5| + C.$$

10. Integrate $\int \frac{2x^3-12x^2+28x-23}{(x-2)^2(x-1)^2}dx$.

Solution: We set $\frac{2x^3-12x^2+28x-23}{(x-2)^2(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$. Multiplying through and solving gives us $A = 0, B = -5, C = 2, D = 1$. Thus, we have

$$\int \frac{2x^3-12x^2+28x-23}{(x-2)^2(x-1)^2}dx = \int \frac{2}{x-2} + \frac{1}{(x-2)^2} - \frac{5}{(x-1)^2}dx$$

$$= 2\ln|x-2| - \frac{1}{x-2} + \frac{5}{x-1} + C.$$

11. Set up the partial fraction decomposition of $\frac{3x^2+1}{(x-1)(x^2+4)^2(x^2+2x+2)^2}$ (you don't have to solve for the coefficients).

Solution: Since x^2+2x+2 is irreducible, we have

$$\frac{3x^2+1}{(x-1)(x^2+4)^2(x^2+2x+2)^2} = \frac{A}{x-1} + + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2} + \frac{Fx+G}{x^2+2x+2} + \frac{Hx+J}{(x^2+2x+2)^2}.$$